ISRAEL JOURNAL OF MATHEMATICS, Vol. 47, Nos. 2-3, 1984

ON L. FEJES TÓTH'S "SAUSAGE-CONJECTURE"

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ABSTRACT

Let k non-overlapping translates of the unit d-ball $B^d \subset E^d$ be given, let C_k be the convex hull of their centers, let S_k be a segment of length $2(k-1)$ and let V denote the volume. L. Fejes Tóth's sausage conjecture says that for $d \geqq 5$

$$V(S_k + B^d) \leqq V(C_k + B^d).$$

In the paper partial results are given.

Let $B_1^d, \cdots, B_k^d$ be k non-overlapping unit d-balls in the euclidean d-space E^d, $d \geqq 2$ (i.e. translates of B^d), let C_k be the convex hull of their centers, let S_k be a segment of length $2(k-1)$, and let V denote the volume.

L. Fejes Tóth conjectured in [3] that for $d \geqq 5$ always

$$V(S_k + B^d) \leqq V(C_k + B^d). \tag{1}$$

Because $S_k + B^d$ forms a "sausage" in E^d, L. Fejes Tóth called this the "sausage-conjecture".

In this paper we give partial results of this problem.

In Theorem 2 we prove it for dim $C_k = 2$. Theorem 2 was found independently by the authors with different proofs. It was the starting point of the paper. In Theorem 3 we show that the "sausage" is at least a relative minimum with respect to the Hausdorff-metric.

In Theorem 4 we investigate relations between the radius of the insphere of C_k and (1).

Theorem 5 shows that (1) holds if dim C_k is small enough compared with d. In [1] a better inequality is shown between dim C_k and d with local methods. But Theorem 5 is of interest because of its global methods.

First in Theorem 1 we show that analogous properties may hold also for some

Received March 1, 1982

of the other quermassintegrals W_i, $i = 0, 1, \cdots, d$, but not for all. For this we need the general Steiner formula (see [4], p. 214)

$$W_i(C_k + B^d) = \sum_{\nu=0}^{d-i} \binom{d-i}{\nu} W_{\nu+i}(C_k), \qquad i = 0, 1, \cdots, d. \tag{2}$$

Instead of the W_i we use the intrinsic volumes (see [5], p. 253) V_{d-i}, which are dimension invariant and defined by

$$\omega_i V_{d-i} = \binom{d}{i} W_i, \qquad i = 0, 1, \cdots, d,$$

where $\omega_i = \pi^{1/2}/\Gamma(i/2+1) = V_i(B^i)$; $V_d = V$, $V_{d-1} = \frac{1}{2}F$ (F is surface area) and $V_0 = 1$. Now (2) can be written

$$V_{d-i}(C_k + B^d) = \sum_{\nu=1}^{d} \binom{\nu}{i} \frac{\omega_\nu}{\omega_i} V_{d-\nu}(C_k), \qquad i = 0, 1, \cdots, d \tag{3}$$

resp. in the very suggestive form

$$V_{d-i}(C_k + B^d) = \sum_{\nu=i}^{d} V_{\nu-i}(B^\nu) V_{d-\nu}(C_k), \qquad i = 0, 1, \cdots, d. \tag{4}$$

For the proofs we need a perhaps well-known inequality

$$\left(\frac{d}{2\pi}\right)^{1/2} < \frac{\omega_{d-1}}{\omega_d} < \left(\frac{d+1}{2\pi}\right)^{1/2}. \tag{*}$$

For a proof of (*) compare [1].

The inequality (*) implies that $f(d) = \omega_{d-1}/\omega_d$ is a strictly increasing function which we use in the proofs without further remarks. For brevity we introduce the

Definition. Let $d \geqq 2$, $i \in [0, d]$, and $1 \leqq n \leqq d - i$ be given. If for each $k > 0$ and each C_k with $\dim C_k \leqq n$

$$V_{d-i}(S_k + B^d) \leqq V_{d-i}(C_k + B^d), \tag{5}$$

we say that the *sausage property* holds in $E^n \subset E^d$ for V_{d-i}, or in short: $\mathrm{SP}(d, d-i, n)$ holds.

So the sausage conjecture means: $\mathrm{SP}(d, d, d)$ holds for $d \geqq 5$. The condition $n \leqq d - i$ is necessary because (5) cannot hold for "large" C_k with $\dim C_k > d - i$.

Theorem 1. *Let $d \geqq 2$ be given. Then*

(a) $\mathrm{SP}(d, d-i, n) \Rightarrow \mathrm{SP}(d+1, d+1-i, n)$ *for each* $i \in [0, d]$ *and* $n \in [1, d-i]$,

(b) $\mathrm{SP}(d, d-i, n) \Rightarrow \mathrm{SP}(d, d+1-i, n)$ *for each* $i \in [1, d]$ *and* $n \in [1, d-i]$,
(c) $\mathrm{SP}(d, d-i, n) \Rightarrow \mathrm{SP}(d-2, d-i, n)$ *for each* $i \in [2, d]$ *and* $n \in [1, d-i]$,
(d) $\mathrm{SP}(d, d-1, n) \Leftrightarrow \mathrm{SP}(d-2, d-2, n)$ *for each* $n \in [1, d-2]$,
(e) $\mathrm{SP}(d, 2, 2)$ *does not hold.*

PROOF.

$$\text{(a)}\qquad \mathrm{SP}(d, d-i, n) \Leftrightarrow \sum_{\nu=i}^{d-1} \binom{\nu}{i} \frac{\omega_\nu}{\omega_i} V_{d-\nu}(C_k) \geqq \binom{d-1}{i} \frac{\omega_{d-1}}{\omega_i} 2(k-1)$$

$$\text{(6)}\qquad \Leftrightarrow \sum_{\nu=i}^{d-1} \binom{\nu}{i} \omega_\nu V_{d-\nu}(C_k) \geqq \binom{d-1}{i} \omega_{d-1} 2(k-1);$$

$$\mathrm{SP}(d+1, d+1-i, n) \Leftrightarrow \sum_{\nu=i}^{d} \binom{\nu}{i} \omega_\nu V_{d+1-\nu}(C_k) \geqq \binom{d}{i} \omega_d 2(k-1)$$

$$\Leftrightarrow \sum_{\nu=i}^{d} \binom{\nu}{i} \omega_\nu \frac{\omega_{d-1}}{\omega_d} \frac{d-i}{d} V_{d+1-\nu}(C_k) \geqq \binom{d-1}{i} \omega_{d-1} 2(k-1).$$

With $V_{d+1-i} = 0$ and ν replaced by $\nu + 1$ this is equivalent to

$$\text{(7)}\qquad \sum_{\nu=i}^{d-1} \binom{\nu+1}{i} \omega_{\nu+1} \frac{\omega_{d-1}}{\omega_d} \frac{d-i}{d} V_{d-\nu}(C_k) \geqq \binom{d-1}{i} \omega_{d-1} 2(k-1).$$

From (6) and (7) it follows that we only need to prove for $\nu = i, \cdots, d-1$:

$$\binom{\nu}{i} \omega_\nu \leqq \binom{\nu+1}{i} \omega_{\nu+1} \frac{\omega_{d-1}}{\omega_d} \frac{d-i}{d}$$

resp.

$$1 \leqq \frac{\nu+1}{d} \frac{d-i}{\nu+1-i} \frac{\omega_{\nu+1}}{\omega_\nu} \frac{\omega_{d-1}}{\omega_d}$$

which is true by (∗).

(b) From (3):

$$V_{d-i}(C_k + B^d) = \sum_{\nu=i}^{d} \binom{\nu}{i-1} \frac{\nu - i + 1}{i} \frac{\omega_\nu}{\omega_i} V_{d-\nu}(C_k)$$

$$= \frac{\omega_{i-1}}{i\omega_i} \sum_{\nu=i}^{d} \binom{\nu}{i-1} (\nu - i + 1) \frac{\omega_\nu}{\omega_{i-1}} V_{d-\nu}(C_k)$$

$$\leqq \frac{d-i}{i} \frac{\omega_{i-1}}{\omega_i} \sum_{\nu=i}^{d} \binom{\nu}{i-1} \frac{\omega_\nu}{\omega_{i+1}} V_{d-\nu}(C_k) + \frac{\omega_d}{i\omega_i} \binom{d}{i-1}$$

$$= \frac{d-i}{i} \frac{\omega_{i-1}}{\omega_i} V_{d-i+1}(C_k + B^d) + \frac{\omega_d}{i\omega_i} \binom{d}{i-1}$$

and "=" for $C_k = S_k$.

(c) From (3):

$$V_{d-i}(C_k+B^d)=\sum_{\nu=i}^{d}\binom{\nu-2}{i-2}\frac{\nu-1}{i-1}\frac{\nu\omega_\nu}{i\omega_i}V_{d-\nu}(C_k)$$

$$=\sum_{\nu=i}^{d}\binom{\nu-2}{i-2}\frac{\nu-1}{i-1}\frac{\omega_{\nu-2}}{\omega_{i-2}}V_{d-\nu}(C_k)$$

$$=\sum_{\mu=i-2}^{d-2}\binom{\mu}{i-2}\frac{\mu+1}{i-1}\frac{\omega_\mu}{\omega_{i-2}}V_{d-2-\mu}(C_k)$$

$$\leqq\frac{d-2}{i-1}\sum_{\mu=i-2}^{d-2}\binom{\mu}{i-2}\frac{\omega_\mu}{\omega_{i-2}}V_{d-2-\mu}(C_k)+\frac{1}{i-1}\binom{d-2}{i-2}\frac{\omega_{d-2}}{\omega_{i-2}}$$

$$=\frac{d-2}{i-1}V_{d-i}(C_k+B^{d-2})+\frac{1}{i-1}\binom{d-2}{i-2}\frac{\omega_{d-2}}{\omega_{i-2}}$$

and " = " for $C_k=S_k$.

(d) $\dim C_k\leqq d-2$ implies $V_{d-1}(C_k)=0$. So from (3) with $i=1$:

$$V_{d-1}(C_k+B^d)=\sum_{\nu=2}^{d}\nu\omega_\nu V_{d-\nu}(C_k)=\pi\sum_{\nu=2}^{d}\omega_{\nu-2}V_{d-\nu}(C_k)$$

$$=\pi\sum_{\mu=0}^{d-2}\omega_\mu V_{d-2-\mu}(C_k)=\pi V_{d-2}(C_k+B^{d-2}).$$

So

$$V_{d-1}(S_k+B^d)\leqq V_{d-1}(C_k+B^d)\Leftrightarrow V_{d-2}(S_k+B^{d-2})\leqq V_{d-2}(C_k+B^{d-2}).$$

(e) Let $k=3$ and $C_k=T^2$ be an equilateral triangle of edge-length 2, so that $V_1(T^2)=3$, $V_2(T^2)=\sqrt{3}$. From (3) we have for $i=d-2$

$$V_2(T^2+B^d)=\sqrt{3}+3(d-1)\frac{\omega_{d-1}}{\omega_{d-2}}+\pi(d-1)\qquad\text{and}$$

$$V_2(S_3+B^d)=4(d-1)\frac{\omega_{d-1}}{\omega_{d-2}}+\pi(d-1),$$

so

$$V_2(T^2+B^d)-V_2(S_3+B^d)=\sqrt{3}-(d-1)\frac{\omega_{d-1}}{\omega_{d-2}}$$

$$=\sqrt{3}-2\pi\frac{\omega_{d-3}}{\omega_{d-2}}<\sqrt{3}-2\pi\frac{\omega_1}{\omega_2}=\sqrt{3}-4<0.$$

THEOREM 2. *If* $\dim C_k = 2$, *then for* $d \geqq 3$:

$$V_d(S_k + B^d) \leqq V_d(C_k + B^d)$$

and equality iff $C_k = S_k$.

PROOF. With $\dim C_k = 2$ we obtain from (3):

$$V_d(C_k + B^d) = \omega_{d-2} V_2(C_k) + \omega_{d-1} V_1(C_k) + \omega_d.$$

If we denote by V'_{ij}, $i = 1, 2$, $j = 1, \cdots, k$ those parts of $V_i(C_k)$ which lie in B^d_j (Fig. 1), then

$$V_d(C_k + B^d) \geqq \sum_{j=1}^{k} (\omega_{d-2} V'_{2j} + \omega_{d-1} V'_{1j}) + \omega_d$$

and equality iff $C_k = S_k$, i.e. if all $V'_{2j} = 0$.

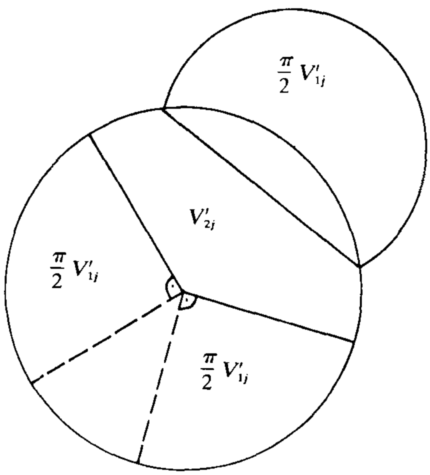

Fig. 1.

For $d \geqq 3$ we have $\omega_{d-1}/\omega_{d-2} \leqq \pi/2$. So

$$V_d(C_k + B^d) \geqq \omega_{d-1} \sum_{j=1}^{k} \left(\frac{\pi}{2} V'_{2j} + V'_{1j} \right) + \omega_d$$

and equality iff $C_k = S_k$. So to prove (1) it suffices to prove

$$2(k-1)\omega_{d-1} + \omega_d \leqq \omega_{d-1} \sum_{j=1}^{k} \left(\frac{2}{\pi} V'_{2j} + V'_{1j} \right) + \omega_d$$

(8)
resp.

$$(k-1)\pi \leqq \sum_{j=1}^{k} \left(V'_{2j} + \frac{\pi}{2} V'_{1j} \right).$$

(8) can be interpreted in aff $C_k = E^2$.

On the left we have the volume of k unit circles $B_j^2 = B_j^d \cap \operatorname{aff} C_k$ minus those parts which come from the cones of the exterior angles of C_k which always give π.

On the right we have $V'_{2j} = V_2(C_k \cap B_j^2)$ and $\frac{1}{2}\pi V'_{1j} = \frac{1}{4}\pi V_1(bdC_k \cap B_j^2)$ (see Fig. 1). So $\frac{1}{2}\pi V'_{1j}$ can be realized by the volume of the half-circle over a segment of $bdC_k \cap B_j^2$ or as two quarter-circles if $bdC_k \cap B_j^2$ contains a vertex of C_k (Fig. 1). So the parts on the right cover those on the left side of (8) and Theorem 2 is proved.

The next theorem shows that the "sausage" is at least a local minimum.

THEOREM 3. *If there is a line* $g \subset E^d$ *with* $\sup_{x \in C_k} \delta(g, x) \leqq \lambda < 1$ *and if* $d \geqq 1 + 2\pi\lambda^2/(1-\lambda^2)$, *then*

$$V_d(S_k + B^d) \leqq V_d(C_k + B^d).$$

PROOF. Let c_i denote the centers of B_i^d, $i = 1, \cdots, k$, and let c'_i be their orthogonal projections onto g. Without restriction let the c'_i be enumerated in the order they lie on g. Then we can define angles φ_i, $i = 1, \cdots, k-1$, by

$$\cos \varphi_i := \frac{\| c'_{i+1} - c'_i \|}{\| c_{i+1} - c_i \|}, \qquad \text{where } 0 \leqq \varphi_i < \frac{\pi}{2} \text{ is assumed.}$$

By our assumption $(1-\lambda^2)^{1/2} \leqq \cos \varphi_i$ and $\sin \varphi_i \leqq \lambda$.

Further let φ and $0 \leqq \psi \leqq \pi/2$ be given by

$$\varphi = \varphi_j := \max_{1 \leqq i \leqq k-1} \varphi_i \quad \text{and} \quad \cos \psi := \frac{\| c'_k - c'_1 \|}{\| c_k - c_1 \|}.$$

By the triangle-inequality we get $\psi \leqq \varphi$.

If $D(C_k)$ denotes the diameter of C_k, then

$$V_1(C_k) \geqq D(C_k) \geqq \| c_k - c_1 \|$$

$$= \frac{1}{\cos \psi} \| c'_k - c'_1 \| = \frac{1}{\cos \psi} \sum_{i=1}^{k-1} \| c'_{i+1} - c'_i \| \geqq \frac{\cos \varphi}{\cos \psi} \cdot 2(k-1).$$

From the monotonicity of V_2 it follows at once that

$$V_2(C_k) \geqq V_2(\operatorname{conv}\{c_1, c_k, c_j, c_{j+1}\}) \geqq D(C_k) \sin(\varphi - \psi)$$

$$\geqq 2(k-1) \cdot (\sin \varphi \cos \varphi - \cos^2 \varphi \tan \psi).$$

This yields

$$\begin{aligned} &V_d(C_k+B^d)-V_d(S_k+B^d) \\ &\geqq \omega_{d-2}V_2(C_k)+\omega_{d-1}(V_1(C_k)-2(k-1)) \\ &\geqq 2(k-1)\omega_{d-1}\left(\frac{\omega_{d-2}}{\omega_{d-1}}(\sin\varphi\cos\varphi-\cos^2\varphi\tan\psi)+\frac{\cos\varphi}{\cos\psi}-1\right) \\ &:=2(k-1)\cdot y(\varphi,\psi), \end{aligned}$$

where $y(\varphi,\psi)$ is defined for $0\leqq\psi\leqq\varphi<\pi/2$.

We want to obtain $y(\varphi,\psi)\geqq 0$. As $y(\varphi,\varphi)=0$ it is sufficient to show

$$\frac{\partial y}{\partial\psi}=\frac{\cos\varphi}{\cos^2\psi}\left(\sin\psi-\frac{\omega_{d-2}}{\omega_{d-1}}\cos\varphi\right)\leqq 0 \qquad \text{for } 0\leqq\psi\leqq\varphi.$$

As $\sin\psi/\cos\varphi\leqq\sin\varphi/\cos\varphi=\tan\varphi$ this is clearly true if

$$\frac{\omega_{d-2}}{\omega_{d-1}}\geqq\sqrt{\frac{d-1}{2\pi}}\geqq\tan\varphi. \tag{9}$$

By our assumption we have $\sin\varphi\leqq\lambda$ and $(d-1)/2\pi\geqq\lambda^2/(1-\lambda^2)$. From this (9) follows and Theorem 3 is proved.

REMARKS. (1) The estimate (9) is actually a generalization of Theorem 3. It is easy to construct polytopes C_k, where the distance to any line is arbitrarily large but φ is arbitrarily small.

(2) For $\lambda\leqq\frac{1}{2}(\varphi\leqq\pi/6)$ the assertion of Theorem 3 holds for $d\geqq 4$, for $\lambda\leqq 1/\sqrt{2}$ $(\varphi\leqq\pi/4)$ it holds for $d\geqq 8$.

(3) Let $\dim C_k=d$. Then $V_d(C_k+B^d)\geqq V_d(S_k+B^d)$ if

$$\sup_{x\in C_k}\delta(g,x)\leqq\frac{\omega_{d-2}}{\sqrt{\omega_{d-2}^2+\omega_{d-1}^2}}\leqq\sqrt{\frac{d-1}{d-1+2\pi}}$$

for a line g.

In contrast to Theorem 3 we now consider a situation in which C_k is not close to any line.

THEOREM 4. *Let* $\dim C_k=n\geqq 2$ *and let the radius* $r(C_k)$ *of the in-sphere of* C_k *be not smaller than* $r>0$. *Then*

$$V_d(C_k+B^d)>mV_d(S_k+B^d) \quad \text{for } d\geqq n-1+n\left(\frac{4}{\pi}\right)^{1/(n-1)}\left(m\left(1+\frac{1}{r}\right)^n\right)^{2/(n-1)}.$$

Here m is an arbitrary positive constant.

PROOF. From a well-known relation ([4], p. 186) we have

$$C_k + B^d \subset \left(1+\frac{1}{r}\right) C_k.$$

If we denote by B^n_ρ an n-ball of radius $\rho \geqq 1$ then

$$V_n(B^n_\rho) = k\omega_n = \rho^n \omega_n \leqq V_n(C_k + B^n) \leqq \left(1+\frac{1}{r}\right)^n V_n(C_k)$$

and by the isoperimetric properties of the V_i, $i = 1, \cdots, n-1$ ([4], p. 278)

$$V_i(B^n_\rho) = \binom{n}{i} \frac{\omega_n}{\omega_{n-i}} \rho^i = \binom{n}{i} \frac{\omega_n}{\omega_{n-i}} k^{i/n} \leqq \left(1+\frac{1}{r}\right)^i V_i(C_k)$$

$i = 1, \cdots, n$.

We only need the cases $i = n-1$ and n, so:

$$\begin{aligned} V_d(C_k + B^d) &= \sum_{i=0}^{n} \omega_{d-i} V_i(C_k) \\ &> \omega_{d-n}\omega_n k \left(1+\frac{1}{r}\right)^{-n} + \omega_{d-n+1}\omega_n \cdot \frac{n}{2} k^{1-1/n} \left(1+\frac{1}{r}\right)^{1-n}. \end{aligned} \tag{10}$$

For

$$\frac{\omega_{d-n}}{\omega_{d-1}} \geqq \frac{2m}{\omega_n} \left(1+\frac{1}{r}\right)^n$$

we obtain

$$\omega_{d-n}\omega_n k \left(1+\frac{1}{r}\right)^{-n} \geqq 2mk\omega_{d-1} > 2m(k-1)\omega_{d-1}$$

and

$$\omega_{d-n+1}\omega_n \frac{n}{2} k^{1-1/n} \left(1+\frac{1}{r}\right)^{1-n} \geqq \frac{\omega_{d-n+1}}{\omega_{d-n}} \frac{\omega_{d-1}}{\omega_d} \frac{n}{2} k^{1-1/n} \left(1+\frac{1}{r}\right) \cdot 2m\omega_d > m\omega_d,$$

i.e.

$$V_d(C_k + B^d) > mV_d(S_k + B^d).$$

So we have to show that (10) holds for the given d and n. For this we remark that

$$\frac{\omega_{d-n}}{\omega_{d-1}} \geqq \left(\frac{d-n+1}{2\pi}\right)^{(n-1)/2} \quad \text{and} \quad \omega_n^{-2/(n-1)} \leqq \pi^{-n/(n-1)} \frac{n}{2}.$$

REMARK. For fixed m and r the condition in Theorem 4 is essentially linear. For instance: for $m=1$, $r(C_k)\geqq 1$ and $n\geqq 9$, then $V_d(C_k+B^d)\geqq V_d(S_k+B^d)$ for $d\geqq 6n-1$.

We are now able to present an estimate with no further restrictions.

THEOREM 5. SP(d,d,n) *holds for* $d\geqq 75n^3$.

PROOF. Making use of Theorem 2 and Theorem 3 we may assume that the distance of C_k from every line is at least $1/\sqrt{2}$ and that $n\geqq 3$. Let $C_k^n:=C_k$ and $\Delta_n:=\Delta(C_k^n)$ denote the minimal width of C_k^n. Then there exists a $(n-1)$-flat H_{n-1} from which C_k^n has the distance $\Delta_n/2$. Let C_k^{n-1} be the orthogonal projection of C_k^n onto H_{n-1} and Δ_{n-1} the minimal width of C_k^{n-1}. Continuing this process we obtain polytopes $C_k^n,\cdots,C_k^1$ with the minimal widths $\Delta_n\leqq\cdots\leqq\Delta_1$.

The distance of C_k to the line H_1 is at most

$$\tfrac{1}{2}\sqrt{\Delta_n^2+\cdots+\Delta_2^2}\leqq\tfrac{1}{2}\Delta_2\sqrt{n-1}$$

and by our assumption we have

$$\frac{\Delta_2}{2}\geqq\frac{1}{\sqrt{2}\sqrt{n-1}}.$$

Let j be the largest index with $\Delta_j/2\geqq 1/\sqrt{2}\sqrt{n-1}$. Then

$$\sin\varphi_i:=\frac{\Delta_i}{2}<\frac{1}{\sqrt{2}\sqrt{n-1}}\qquad\text{for } i>j.$$

The distances of the vertices of C_k^j and of the other projections of the centers of $B_1^d,\cdots,B_k^d$ onto H_j are at least

$$2\cdot\prod_{i=j+1}^{n}\cos\varphi_i\geqq 2\cdot\left(\frac{2n-3}{2n-2}\right)^{(n-1)/2}.$$

Defining

$$P_k^j:=\left(\frac{2n-2}{2n-3}\right)C_k^j$$

the distances become at least 2. Now we have:

$$r(P_k^j)=\left(\frac{2n-2}{2n-3}\right)^{(n-1)/2}\cdot r(C_k^j)$$

$$\geqq \left(\frac{2n-2}{2n-3}\right)^{(n-1)/2} \cdot \Delta_j \cdot \begin{cases} \dfrac{\sqrt{j+2}}{2j+2} & \text{for } j \text{ even} \\ \dfrac{1}{2\sqrt{j}} & \text{for } j \text{ odd} \end{cases} \qquad \text{(see [2])}$$

(11)
$$\geqq \left(\frac{2n-2}{2n-3}\right)^{(n-1)/2} \cdot \frac{1}{\sqrt{2}\sqrt{n-1}\cdot\sqrt{j+1}}.$$

Furthermore, we obtain by easy Steiner-symmetrization and by the homogeneity of the V_i:

$$V_i(C_k) \geqq V_i(C_k^j) = \left(\frac{2n-3}{2n-2}\right)^{i(n-1)/2} V_i(P_k^j)$$

$$\geqq \left(\frac{2n-3}{2n-2}\right)^{j(n-1)/2} V_i(P_k^j) \qquad \text{for } i = 0, \cdots, j$$

which yields

$$V_d(C_k + B^d) \geqq \left(\frac{2n-3}{2n-2}\right)^{j(n-1)/2} V_d(P_k^j + B^d)$$

(12)
$$\geqq V_d(S_k + B^d)$$

if

$$V_d(P_k^j + B^d) \geqq \left(\frac{2n-2}{2n-3}\right)^{j(n-1)/2} V_d(S_k + B^d).$$

We remark that these relations remain true for the case $j = n$.

Using Theorem 4 and (11) we obtain that (12) holds for

$$d \geqq j - 1 \cdot \left(\frac{4}{\pi}\right)^{1/(j-1)} \cdot \left(\sqrt{2(n-1)}\sqrt{j+1} + \left(\frac{2n-2}{2n-3}\right)^{(n-1)/2}\right)^{2j/(j-1)}.$$

For $2 \leqq j \leqq n$ and $n \geqq 3$ this can be estimated as follows:

$$j - 1 + j \cdot \left(\frac{4}{\pi}\right)^{1/(j-1)} \cdot \left(\sqrt{2(n-1)}\sqrt{j+1} + \left(\frac{2n-2}{2n-3}\right)^{(n-1)/2}\right)^{2j/(j-1)}$$

$$\leqq n - 1 + n\frac{4}{\pi} \cdot (\sqrt{2(n-1)}\sqrt{j+1} + e^{1/4})^{2j/(j-1)}.$$

As

$$\frac{\partial}{\partial j}(\sqrt{2(n-1)}\sqrt{j+1} + e^{1/4})^{2j/(j-1)} \leqq 0$$

in $[2, n]$ we may continue

$$\leqq n - 1 + n \cdot \frac{4}{\pi} \left(\sqrt{6}\sqrt{n-1} + e^{1/4}\right)^4$$

$$\leqq 75n^3$$

which proves our theorem.

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